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LOCAL INTEGRAL EQUATIONS FOR TRANSIENT HEAT CONDUCTION IN FUNCTIONALLY GRADED MATERIALS

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In this paper, we employ a meshless point interpolation approximation for the field variable in the numerical implementation of local integral equations (LIEs) applied to 2-d transient heat conduction problems in functionally graded materials (FGMs). Then, the approximation is inherently consistent and the system matrix is sparse like in standard finite element formulation. In contrast to the weak form FEM formulations, two kinds of the integral equation formulation are applied. The time evolution is treated either by using the θ-method or the Laplace transform technique supplemented with the Stehfest's inversion algorithm. The method is sufficiently simple and quite general. The accuracy of the proposed method is tested on several examples. Comparison is made with the numerical results by using standard domain elements in numerical implementation of the LIEs.

Key Words: Continuous Non-homogeneity, Transient Problems, Meshless Interpolation

1. Introduction

With recent development and research of functionally graded materials (FGMs), problems in non-homogeneous media have generated new interest. In FGMs, the composition and the volume fraction of the FGM constituents vary gradually, giving a non-uniform microstructure with continuously graded macro-properties. Although the versatile and well established finite element method (FEM) and the boundary integral equation method (BIEM) or boundary element method (BEM) are basically applicable to such problems, some difficulties may occur in the numerical analysis.

The main goal of this paper is to propose and test a sufficiently simple numerical technique appropriate for solution of transient boundary-initial problems in media with arbitrary variation of material coefficients. In this study, we confine to 2-d transient heat conduction problems.

2. Formulation of the problem

The transient heat conduction equation is given by

$$\left(\lambda(\mathbf{x})u_{,i}(\mathbf{x},t)\right)_{,i} - \rho(\mathbf{x})c\frac{\partial u(\mathbf{x},t)}{\partial t} = -w(\mathbf{x},t), \qquad (1)$$

in $\Omega\times[0,T]$, where $u(\mathbf{x},t)$ is the temperature field, $\lambda(\mathbf{x})$ is the thermal conductivity, $\rho(\mathbf{x})$ is the mass density, c is the specific heat per unit mass and $w(\mathbf{x},t)$ is the density of heat sources. In the absence of heat sources, the governing equation converts to the diffusion equation with the diffusivity coefficient $\kappa(\mathbf{x}) = \lambda(\mathbf{x})/\rho(\mathbf{x})c$. Without loss of generality, we assume $w(\mathbf{x},t) = 0$ in this paper. In an initial-boundary value problem, one should prescribe the initial value of the potential field

$$u(\mathbf{x},0) = v(\mathbf{x}) \text{ at } \mathbf{x}\in\Omega, \qquad (2)$$

and the boundary values of the potential or the flux

$$u(\boldsymbol{\eta},t) = \tilde{u}(\boldsymbol{\eta},t) \quad \text{at} \quad \boldsymbol{\eta}\in\partial\Omega_D,$$

$$-n_i(\boldsymbol{\eta})\lambda(\boldsymbol{\eta})u_{,i}(\boldsymbol{\eta},t) = \tilde{q}(\boldsymbol{\eta},t) \quad \text{at} \quad \boldsymbol{\eta}\in\partial\Omega_N, \qquad (3)$$

where $\partial\Omega_D$ and $\partial\Omega_N$ are the Dirichlet and Neumann parts of the global boundary $\partial\Omega$ of the domain Ω, with $\partial\Omega_D\cap\partial\Omega_N = 0$. The tilde denotes the boundary densities prescribed by boundary conditions.

3. Local integral equations with singular kernels

Since the governing equation is given by the partial differential equation with variable coefficients, it is impossible to find the fundamental solution of Eq. (1) in a closed form in general. Nevertheless, one can use the fundamental solution of the Laplace operator, which in two-dimensional case is given as

$$G(r) = -\frac{1}{2\pi\lambda^c}\ln r\,, \tag{4}$$

with λ^c being an arbitrary constant.

Let $\mathbf{y}^c$ be an arbitrary interior point in Ω and Ω^c be an arbitrary subdomain of the domain Ω, with $\mathbf{y}^c \in \Omega^c \subset \Omega$. From the governing equation (1) with $w(\mathbf{x},t) = 0$, we have the integral identity

$$\int_{\Omega^c}\left[\left(\lambda(\mathbf{x})u_{,i}(\mathbf{x},t)\right)_{,i} - \rho(\mathbf{x})c\frac{\partial u(\mathbf{x},t)}{\partial t}\right]G(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) = 0 \tag{5}$$

Applying the Gauss divergence theorem to (5), we can derive the local integral equation

$$\int_{\partial\Omega^c}\lambda(\boldsymbol{\eta})n_i(\boldsymbol{\eta})u_{,i}(\boldsymbol{\eta},t)G(\boldsymbol{\eta}-\mathbf{y}^c)d\Gamma(\boldsymbol{\eta}) - \int_{\Omega^c}\lambda(\mathbf{x})u_{,i}(\mathbf{x},t)G_{,i}(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) - c\int_{\Omega^c}\rho(\mathbf{x})\frac{\partial u}{\partial t}(\mathbf{x},t)G(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) = 0\,, \tag{6}$$

which will be referred to as the LIE of the 1st kind. The second term in (6) can be further rearranged. For this purpose, we use the notations $\lambda^c = \lambda(\mathbf{x}^c)$, $\tilde{\lambda}(\mathbf{x}) = \lambda(\mathbf{x}) - \lambda^c$. Then, using the Gauss divergence theorem, we rewrite Eq. (6) as

$$u(\mathbf{y}^c,t) + \int_{\partial\Omega^c}\left[u(\boldsymbol{\eta},t)\lambda^c G_{,i}(\boldsymbol{\eta}-\mathbf{y}^c) - \lambda(\boldsymbol{\eta})u_{,i}(\boldsymbol{\eta},t)G(\boldsymbol{\eta}-\mathbf{y}^c)\right]n_i(\boldsymbol{\eta})d\Gamma(\boldsymbol{\eta}) + \int_{\Omega^c}\tilde{\lambda}(\mathbf{x})u_{,i}(\mathbf{x},t)G_{,i}(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) + c\int_{\Omega^c}\rho(\mathbf{x})\frac{\partial u}{\partial t}(\mathbf{x},t)G(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) = 0\,. \tag{7}$$

This equation is formally equivalent to equation (6) and will be referred to as the LIE of the 2nd kind. One worthwhile difference consists in fact that the domain integral of temperature gradients is involved in Eq. (6) even if the medium is homogeneous. After spatial discretisation, the derived LIEs convert to a set of ordinary differential equations which can be solved by using the one-time-step θ-method.

Without going into details, we present the derived LIE in the Laplace transform (LT). The LIE of the 1st kind is given as

$$\int_{\partial\Omega^c}\lambda(\boldsymbol{\eta})n_i(\boldsymbol{\eta})\bar{u}_{,i}(\boldsymbol{\eta},p)G(\boldsymbol{\eta}-\mathbf{y}^c)d\Gamma(\boldsymbol{\eta}) - \int_{\Omega^c}\lambda(\mathbf{x})\bar{u}_{,i}(\mathbf{x},p)G_{,i}(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) - cp\int_{\Omega^c}\rho(\mathbf{x})\bar{u}(\mathbf{x},p)G(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) = -c\int_{\Omega^c}\rho(\mathbf{x})v(\mathbf{x})G(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x})\,, \tag{8}$$

and the LIE of the 2nd kind as

$$\bar{u}(\mathbf{y}^c,p) + \int_{\partial\Omega^c}\left[\bar{u}(\boldsymbol{\eta},p)\lambda^c G_{,i}(\boldsymbol{\eta}-\mathbf{y}^c) - \lambda(\boldsymbol{\eta})\bar{u}_{,i}(\boldsymbol{\eta},p)G(\boldsymbol{\eta}-\mathbf{y}^c)\right]n_i(\boldsymbol{\eta})d\Gamma(\boldsymbol{\eta}) + \int_{\Omega^c}\tilde{\lambda}(\mathbf{x})\bar{u}_{,i}(\mathbf{x},p)G_{,i}(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) + cp\int_{\Omega^c}\rho(\mathbf{x})\bar{u}(\mathbf{x},p)G(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x}) = c\int_{\Omega^c}\rho(\mathbf{x})v(\mathbf{x})G(\mathbf{x}-\mathbf{y}^c)d\Omega(\mathbf{x})\,. \tag{9}$$

The over bar denotes the LT of the time dependent quantity. In the Laplace transform approach, the numerical inversion of the LT is a key issue since it is an ill-posed problem. Various Laplace inversion algorithms are available in literature. The advantages and deficiencies of some algorithms were pointed out by Maillet et al [1], and Davies and Martin [2] made a critical study of the various algorithms. Regarding to a good experience [4] with the Stehfest's algorithm [3], we have also used this algorithm in the present analysis.

4. Integral form of the balance equation (IFBE)

The general physical balance principles of any continuum theory take the form of integral equations. The governing equations (or field equations) take the form of differential equations which are derived from these integral principles with taking into account that they hold for all arbitrary but small material domains. Following a reverse path, we can get the integral form of the balance principle by integrating the governing equation over $\Omega^c \subset \Omega$. Thus, integrating Eq. (1), we have

$$\int_{\Omega^c}\left[\left(\lambda(\mathbf{x})u_{,i}(\mathbf{x},t)\right)_{,i} - \rho(\mathbf{x})c\frac{\partial u(\mathbf{x},t)}{\partial t}\right]d\Omega(\mathbf{x}) = 0\,, \tag{10}$$

hence,

$$\int_{\partial\Omega^c}\lambda(\boldsymbol{\eta})n_i(\boldsymbol{\eta})u_{,i}(\boldsymbol{\eta},t)d\Gamma(\boldsymbol{\eta}) - c\int_{\Omega^c}\rho(\mathbf{x})\frac{\partial u}{\partial t}(\mathbf{x},t)d\Omega(\mathbf{x}) = 0 \tag{11}$$

which will be referred to as the integral form of the balance equation (IFBE).

Eventually, in the LT approach Eq. (11) becomes

$$\int_{\partial\Omega^c} \lambda(\eta) n_i(\eta) \overline{u}_{,i}(\eta, p) d\Gamma(\eta) -$$

$$cp \int_{\Omega^c} \rho(\mathbf{x}) \overline{u}(\mathbf{x}, p) d\Omega(\mathbf{x}) = -c \int_{\Omega^c} \rho(\mathbf{x}) v(\mathbf{x}) d\Omega(\mathbf{x}) . \quad (12)$$

5. Meshless interpolations of the temperature field

Let the field variable $u(\mathbf{x},t)$ or its LT $\overline{u}(\mathbf{x}, p)$ in $\Omega \cup \partial\Omega$ be approximated within a subdomain $\Omega_s \subset (\Omega \cup \partial\Omega)$. Assuming the dimensionalities of both Ω_s and Ω to be the same, we have a domain type approximation. In each meshless approximation technique, the aim is to create shape functions for approximation of the field variable within the subdomain Ω_s using only nodes scattered arbitrarily in the analyzed domain without any predefined mesh to provide connectivity of the nodes.

Assuming a finite series representation of the field variable in a subdomain Ω_s^q surrounding the nodal point $\mathbf{x}^q$, we have the approximated field given as

$$u(\mathbf{x})\big|_{\Omega_s^q} = \sum_{a=1}^{N} B^a(\mathbf{x}) c^a(\mathbf{x}^q) \quad (13)$$

where $u(\mathbf{x})$ stands for $u(\mathbf{x},t)$ or $\overline{u}(\mathbf{x}, p)$, $B^a(\mathbf{x})$ are the basis functions defined in the Cartesian coordinate space, N is the number of nodes in the support domain of the point $\mathbf{x}^q$, and $c^a(\mathbf{x}^q)$ are the expansion coefficients corresponding to that point. As regards the choice of the basis functions, we shall consider monomials $P^k(\mathbf{x}) \in \left\{1, x_1, x_2, x_1 x_2, x_1^2, x_2^2\right\}$ for $k = 1,\ldots,6$ as polynomial basis functions and multiquadrics (MQ)

$$R^n(\mathbf{x}) = \left(\left|\mathbf{x} - \mathbf{x}^n\right|^2 + \mu^2\right)^{m/2}$$

as the radial basis functions (RBF). Then, the point approximations become true Point Interpolation Methods [5], since the derived shape functions possess the Kronecker delta property [6]. Three different combinations of these basis functions have been utilized in the numerical implementation of the local integral equations in stationary problems [6]. Bearing in mind the result of the study on numerical stability and patch tests [6], we confine to *PBF + MQ interpolation approach* with $M < N$

$$u(\mathbf{x})\big|_{\Omega_s^q} = \sum_{k=1}^{N} R^{n(q,k)}(\mathbf{x}) \alpha^{(q,k)} + \sum_{k=1}^{M} P^k(\mathbf{x}) \beta^{(q,k)}$$

The approximated field can be expressed in terms of the shape function $\varphi^{(q,j)}(\mathbf{x})$ as

$$u(\mathbf{x})\big|_{\Omega_s^q} = \sum_{j=1}^{N} u(\mathbf{x}^{n(q,j)}) \varphi^{(q,j)}(\mathbf{x}) , \quad (14)$$

in which $n(q, j)$ is the global number of the j^{th} nearest nodal point of N supporting nodes corresponding to $\mathbf{x}^q$. For more details, we refer the reader to Ref.[6].

Finally, the numerical results will be compared also with the results obtained by using the standard quadrilateral elements with quadratic interpolation (QQE approach). For their numerical implementation, we refer to Ref. [7].

6. Numerical examples

In order to test the proposed numerical method, we have considered examples for which analytical solutions are available. The considered domain is a square $L \times L$ with the Dirichlet boundary conditions on both the bottom and top sides, while the Neumann conditions are assumed on the lateral sides. Although the analytical solutions are available for several events of unidirectional variation of material coefficients, we confine the presented results to exponential gradation of both the heat conduction and mass density as

$\lambda(\mathbf{x}) = \lambda_o e^{\delta x_2 / L}$, $\rho(\mathbf{x}) = \rho_o e^{\delta x_2 / L}$, in this paper.

In the convergence study for the accuracy of the numerical results with respect to increasing the density of nodal points, we have used the average % error defined as

$$\text{APE}_t = 100 \sqrt{\sum_{a=1}^{N_t} \left[u^c(\mathbf{x}^a) - u^{ex}(\mathbf{x}^a)\right]^2 / \sum_{a=1}^{N_t} \left[u^{ex}(\mathbf{x}^a)\right]^2}$$

where $u^c(\mathbf{x}^a)$ and $u^{ex}(\mathbf{x}^a)$ stand for the computed and exact nodal values, respectively, with N_t being the total number of nodes on closed domain $\Omega \cup \partial\Omega$.

The numerical computations are carried out by using three different kinds of integral equations (LIE of the 1st and 2nd kinds, and IFBE) combined with various domain-type approximations. Recall that there is no difference between the results by the LIE of the 1st and 2nd kinds in case of non-homogeneous media. In this paper, we assume the number of the radial basis functions $N = 16$, with their shape parameters being $m = -2$ and $\mu = 2h$, where h is the shortest distance of any two nodal points. The polynomial basis is given by $M = 6$ monomials.

Owing to the convergence study and comparison of the results by using various domain-type approximations, we assume only uniform distributions of nodal points with h/L being a parameter characterizing the density of such distributions.

Example 1

In this example, we assume the following initial and boundary conditions

$v(\mathbf{x}) = v_o = 1$, $\tilde{u}(x_1,0,t) = u_o = 1$, $\tilde{u}(x_1,L,t) = u_L = 20$, $\tilde{q}(0,x_2,t) = -\tilde{q}(L,x_2,t) = 0$.

and the material parameters have been chosen as: $\lambda_o = 1$, $\rho_o = 1$, $c = 1$, $\delta = 3$, $L = 1$.

Fig. 1 shows the comparison of numerically computed temperature field with analytical values at two time instants by using both the LIE and IFBE in combinations with the QQE-interpolation and the PBF+MQ interpolation. The numerical data correspond to the use of the implicit Galerkin's scheme from the family of one-time-step θ-methods, but the difference from the results by using LT-approach is not visible. Therefore, we confine to give the CPU-times in the case of LT-approach. In both the time instants t=0.02 and t=1, they are the same, since only 1 step is used. In the case of QQE-interpolation it is 7.8 [sec], while in the PIM it is 84 [sec]. Note that t=0.02 is a very early time when the initial value of the temperature is not changed in a half of the specimen, while t=1 is the late time instant when the temperature distribution represents stationary one as can be seen from Fig. 2.

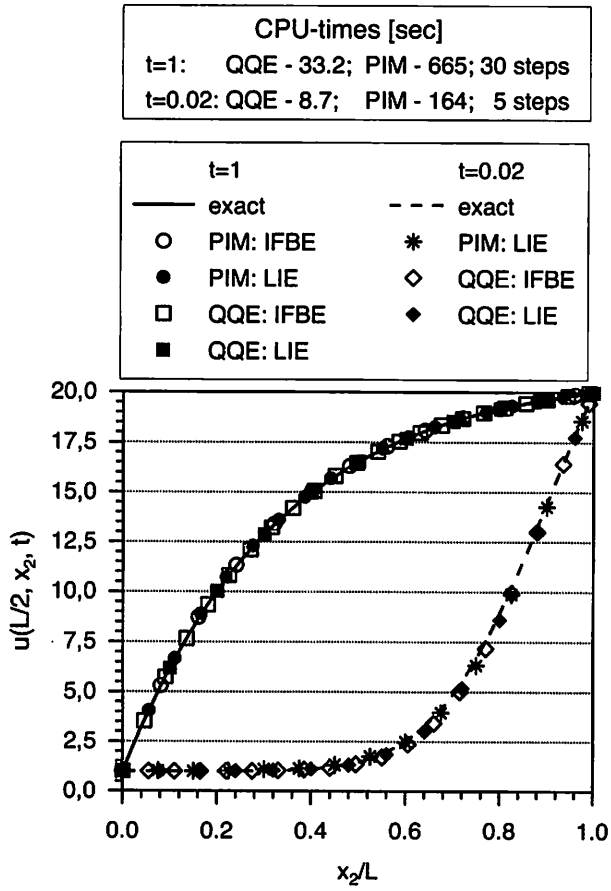

Fig.1 Distribution of temperature along vertical direction at two time instants t=0.02 and t=1 using θ-method

In all presented numerical results, we have used QQE-interpolation with 341 nodes (100 quadrilateral quadratic elements) and PIM with 121 nodes.

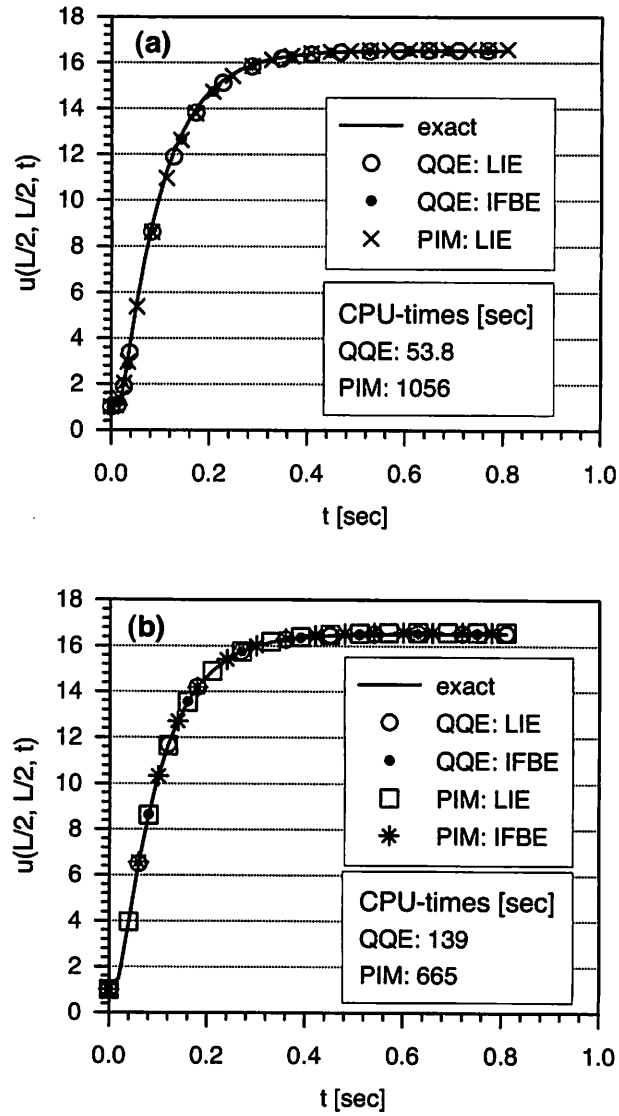

Fig.2 Time evolution of the temperature at midpoint using (a) θ-method with 50 time steps; (b) LT-method with 10 LT-parameters at each time instant. The CPU-time in LT-approach corresponds to 30 time instants.

The perfect agreement of all numerical results with the exact ones can be observed, but it should be stressed that the IFBE in combination with PIM fails at early time instants when LT-approach is used. This can be seen in Fig.3. Although the accuracy for IFBE combined with PIM and presented in Fig. 4 is acceptable, there has been observed instability with respect to the choice of the time steps. This instability is expected to be removed by modification of the selection of support nodes for RBFs with incorporating also more distant nodes into account.

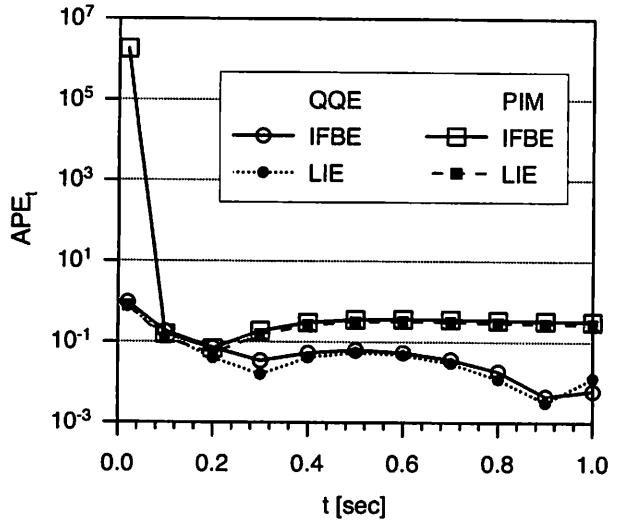

Fig.3 Dependence of accuracy on the final time instant using LT -approach

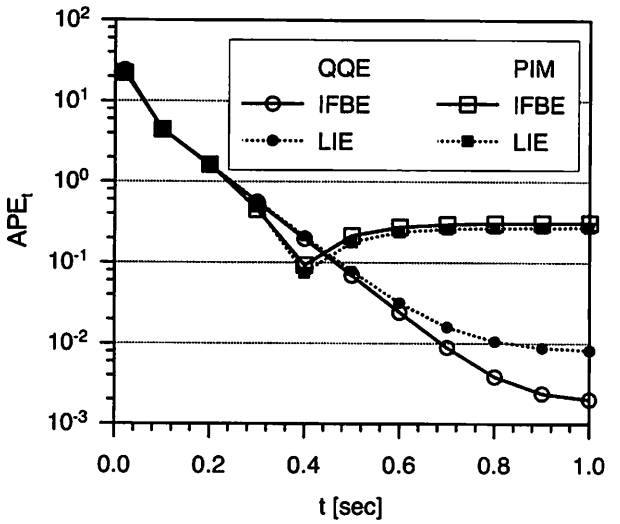

Fig.4 Dependence of accuracy on the final time instant using θ-method with 50 time steps

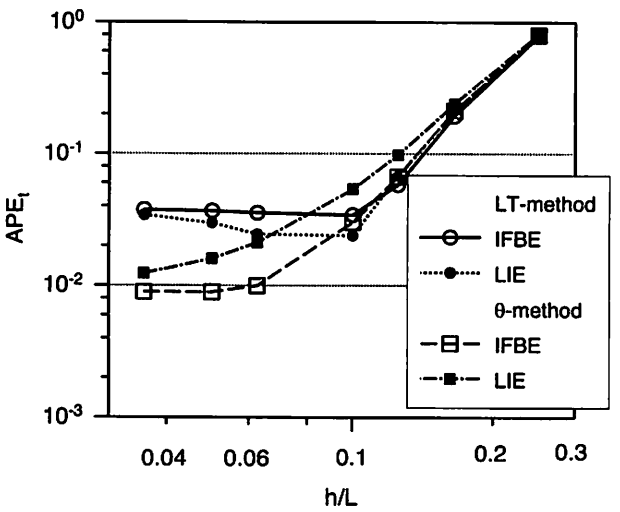

Fig.5 Convergence of numerical results at time instant t=0.7 with increasing the density of nodal points in QQE-interpolation

Finally, Figs. 5 and 6 show some results of the convergence study.

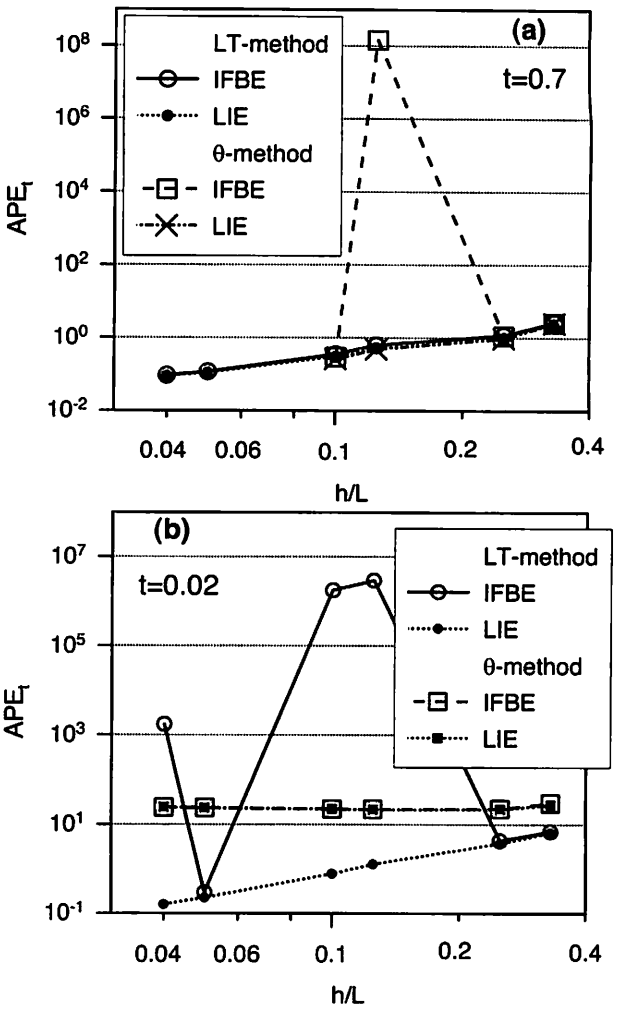

Fig.6 Failure of the IFBE combined with PIM at certain nodal point distributions: (a) in θ-method; (b) at early time instants in LT-approach

Example 2

In this example, the time-dependent boundary condition is prescribed on one face of the square domain, while the other faces are insulated. The initial and the boundary conditions are

$$v(\mathbf{x}) = v_o = 0\,, \quad \tilde{u}(x_1, L, t) = 10t\,, \quad \tilde{q}(x_1, 0, t) = 0\,,$$

$$\tilde{q}(0, x_2, t) = -\tilde{q}(L, x_2, t) = 0\,.$$

As compared with *Ex.1*, the material parameters are changed as follows $\lambda_o = 5\,, \delta \in \{0,1,3\}$.

In the numerical calculations, we have used the same distributions of nodal points as in the previous example. The numerical results shown in Fig.7 have been obtained by using 50 time steps and the CPU-times corresponding to the QQE-interpolation and the PIM are 23 [sec] and 65 [sec], respectively. The case of $\delta = 0$ represents a homogeneous medium. Since the analytical solution is not available, the present numerical solution for a

material gradient parameter $\delta = 3$ is compared with the BEM solution of the work [4].

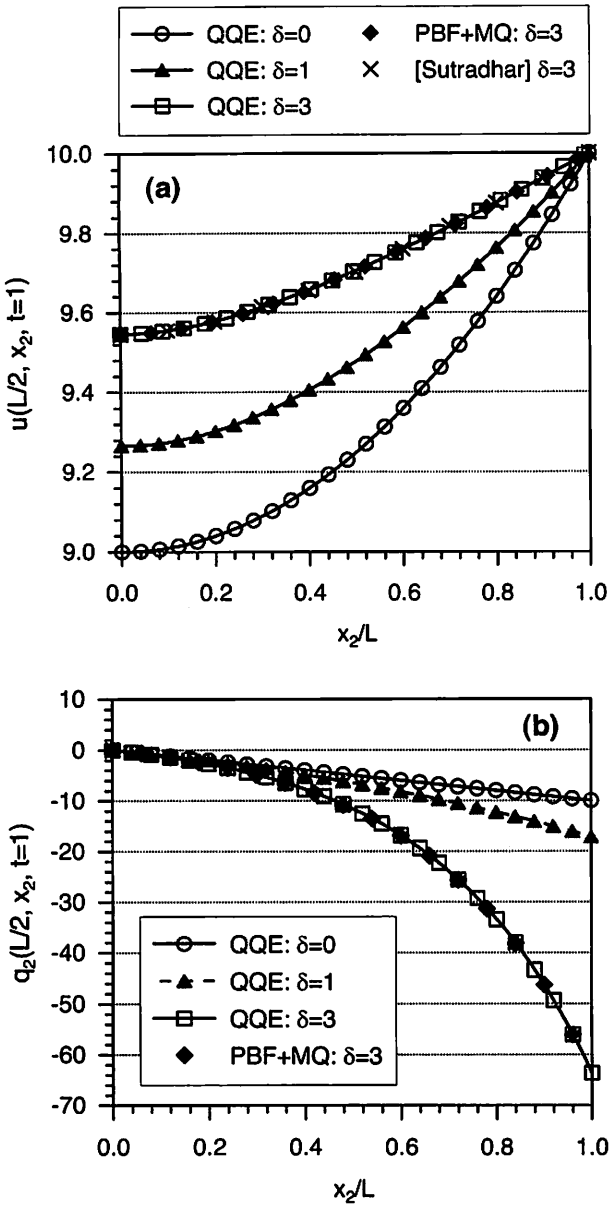

Fig.7 Distribution of (a) temperature, (b) flux at time instant t=1 along the vertical line for various gradient parameters by using LIE and θ-method approach

7. Conclusions

New computational techniques are proposed for numerical solution of 2-d transient heat conduction problems in non-homogeneous media. The techniques consist in combinations of three different kind of integral equations with two domain-type interpolations of field variable. One of the interpolation approaches utilizes the standard quadrilateral quadratic serendipity elements. The other is a meshless point interpolation method based on the use of the polynomial functions and the radial functions. The integral equations are considered on local subdomains surrounding interior nodes and are non-singular. Using meshless interpolations, we can easily deal with domain integrals due to material non-homogeneity and the problem of mesh generation is avoided. The time variable is treated either by using the θ-method or the Laplace transform technique with the Stehfest's numerical inversion.

The proposed method is quite general as regards the non-homogeneity and a satisfactory accuracy has been achieved in numerical tests. Note that the approach based on the use of the IFBE + PIM exhibits some instability with respect to the choice of time steps in θ-method and it fails in some nodal points distributions at very early time instants in the LT-approach.

The method is open for the use of other meshfree approximations. The extension of the method to three dimensions as well as to other applications is straightforward.

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